1028 MMP

Thursday, October 27, 2022 14:08

Goal.

- Thm 1.8: $d, r \in \mathbb{Z}_{>0}$ $\quad \varepsilon \in \mathbb{R}_{>0}$. Then $\exists\, t = t(d, r, \varepsilon) \in \mathbb{R}_{>0}$ s.t. if we assume
  - (X, B) is a projective ε-lc pair $\quad \dim X = d$
  - A is a very ample div. on X w/ $A^d \le r$
  - $A - B$ is pseudo-effective
  - $M \ge 0$ is $\mathbb{R}$-Cartier $\mathbb{R}$-div s.t. $A - M$ is ps-eff.

  then $\underline{\operatorname{lct}(X, B; |M|_{\mathbb{R}}) \geq \operatorname{lct}(X, B; |A|_{\mathbb{R}}) \geq t}$

- Thm 1.6: $d \in \mathbb{Z}_{>0}$ $\quad \varepsilon \in \mathbb{R}_{>0}$. Then $\exists\, t = t(d, \varepsilon) \in \mathbb{R}_{>0}$ s.t. if we assume
  - (X, B) is proj. ε-lc $\quad \dim X = d$
  - $A := -(K_X + B)$ nef & big

  then $\operatorname{lct}(X, B; |A|_{\mathbb{R}}) \geq t$

- Thm 1.1 (BAB): $d \in \mathbb{Z}_{>0}$ $\quad \varepsilon \in \mathbb{R}_{>0}$

  The proj. varieties X s.t.
  - $\dim X = d$
  - (X, B) ε-lc for some boundary B
  - $-(K_X + B)$ nef & big

  form a bounded family.

- $1.8_d + 1.1_{\le d-1} \xRightarrow{\text{Lem 3.2}} 1.6_d$
- $1.6_d + 1.1_{\le d-1} \Rightarrow 1.1_d$
- $5.7 + 5.11 \Rightarrow 1.8_d$

  (Last time) $\nwarrow$ $\quad$ $\Uparrow$ 1.8_{d-1}

- Prop 5.7: $d, r, n \in \mathbb{Z}_{>0}$ $\quad \varepsilon \in \mathbb{R}_{>0}$ Assume Thm $1.8_{\le d-1}$

  Then $\exists\, q = q(d, r, n, \varepsilon) \in \mathbb{Z}_{>0}$ s.t. if assume
  - (X, B) proj. ε-lc $\quad \dim X = d$
  - A very ample on X $\quad A^d \le r$
  - $L \ge 0$ $\mathbb{R}$-div on X
  - $\Lambda \ge 0$ $\mathbb{Q}$-div on X $\quad n\Lambda$ integral.
  - $A - B$, $A - L$, $A - \Lambda$ ample
  - (X, Λ) lc near $x \in X$
  - T is an lc place of (X, Λ) w/ $c_X(T) = \overline{x}$
  - $a(T, X, B) \le 1$

  then for $\forall$ $\nu : U \xrightarrow{} X$ (with $T \subset U$) we have

  $\underline{\operatorname{mult}_T(\nu^* L) \le q}$.

- Prop 5.11: $d, r \in \mathbb{Z}_{>0}$ $\varepsilon \in \mathbb{R}_{>0}$ Assume Thm $1.8_{\le d-1}$

  Then $\exists$ $\begin{matrix} n = n(d, r, \varepsilon) \\ m = m(d, r, \varepsilon) \end{matrix} \in \mathbb{Z}_{>0}$ $\quad 0 < \varepsilon' = \varepsilon'(d, r, \varepsilon) < \varepsilon$ s.t. if
  - (X, B) proj. ε-lc $\quad \dim X = d$
  - A very ample on X $\quad A^d \le r$
  - $L \ge 0$ $\mathbb{R}$-div on X
  - $A - B$ $\quad A - L$ ample
  - $(X, B + tL)$ ε'-lc for some $t \in (0, r)$
  - → $a(T, X, B + tL) = \varepsilon'$ for some prime div T / X
  - $c_X(T) = \overline{x}$ for some $x \in X$

  Then $\exists$ $\mathbb{Q}$-Cart. $\mathbb{Q}$-div $\Lambda \ge 0$ s.t.

  ① $n\Lambda$ integral
  ② $mA - \Lambda$ ample
  ③ (X, Λ) lc near x
  ④ T is an lc place of (X, Λ).

- Proof (8 steps) $\quad$ WMA $d > 1$

  $\varepsilon' = \varepsilon'(d, \varepsilon) < \varepsilon$. $\quad$ 1 - gap of lct of reduced pairs in dim d

  (Y, S) $\quad (Y, (1 - \varepsilon')S)$ lc $\Rightarrow (Y, S)$ is lc. (ACC of lct).

  Step 1. $\exists\, v = v(d, r, \varepsilon)$ s.t.

  $(X, (1 + v)(B + tL))$ is $\frac{\varepsilon'}{2}$-lc outside finitely many closed pts.

  $(r + 1)A - B - tL = \underline{A - B} + t\,\underline{(A - L)} + \overset{>0}{(r - t)}A$ ample

  WMA $A - B - tL$ is ample.

  $H \in |A|$ general $\quad$ By $1.8_{\le d-1}$

  $\exists\, v = v(d, r, \varepsilon') = v(d, r, \varepsilon)$ s.t.

  $(H, B_H + tL_H + 2v(B_H + tL_H))$ klt

  $\xRightarrow{\text{inv. adj}} (X, H + (1 + 2v)(B + tL))$ is plt near H.

  $\Rightarrow \underline{(X, (1 + 2v)(B + tL))}$ klt near H

  $\Downarrow$ outside fin. many closed pts.

  $\Rightarrow \underline{(X, (1 + v)(B + tL))} \quad \frac{\varepsilon'}{2}$-lc.

  Step 2 $T \subset W \xrightarrow{\psi} (X, B + tL)$

  $\Gamma_W = \underline{(1 + v)(\widetilde{B} + t\widetilde{L})} + \underline{(1 - \tfrac{\varepsilon'}{4})\sum E_i} + \underline{(1 - \varepsilon')T}$

  E_i exc div on W / X $\quad \ne T$.

  $c_i = a(E_i, X, B + tL) \ \underline{\ge \varepsilon'}$

  $\varepsilon' = a(T, X, B + tL)$

  $\underline{K_W + \Gamma_W = \psi^*(K_X + B + tL) + v(\widetilde{B} + t\widetilde{L})} + F$

  $F = \underset{\ge 0}{\sum (c_i - \tfrac{\varepsilon'}{4})} E_i$ $\quad$ exc / X support doesn't $\supset T$.

  $c_i' = a(E_i, X, (1 + v)(B + tL))$

  $a' = a(T, X, (1 + v)(B + tL))$

  $\underline{K_W + \Gamma_W = \psi^*(K_X + (1 + v)(B + tL)) + G}$

  $G = \sum (c_i' - \tfrac{\varepsilon'}{4}) E_i + (a' - \varepsilon')T$

  Step 1 $\Rightarrow$ if $c_X(E_i)$ is not a closed pt on X

  $\underline{\operatorname{mult}_E(G) > 0}$.

  Step 3 $\quad \underline{K_W + \Gamma_W - \text{MMP} / X \quad \text{does not contract } T}$.

  - (K_W, Γ_W) is klt.

  $W \dashrightarrow Y' \xrightarrow{\phi'} X$

  Step 2. $\begin{cases} \phi' \text{ is small outside finitely many closed pts on } X. \\ K_W + \Gamma_W \sim_{\mathbb{R}} v(\widetilde{B} + t\widetilde{L}) + F / X \leftarrow T \text{ not a comp't} \end{cases}$

  $\Rightarrow T$ not contracted.

  $A_{Y'} = \phi'^* A$

  bpf + bddness of length of extremal ray

  $\Rightarrow \underline{\boxed{K_{Y'} + \Gamma_{Y'} \quad 2d A_{Y'}} \text{ is semi-ample.}}$

  Step 4 $\quad \exists\, Y \xrightarrow{\phi} X$ $\quad$ s.t. $-T_Y$ ample $/ X$

  extract T and take ample model.

  $W \dashrightarrow \underline{Y' \dashrightarrow Y \quad \text{is a bir contraction}}$

  Step 5. $\underline{K_Y + \Gamma_Y + 3d A_Y}$ ample.

  - $K_Y + \Gamma_Y \sim_{\mathbb{R}} (a' - \varepsilon')T_Y / X \quad \mathbb{R}$-Cartier.

    $\Rightarrow \mathbb{R}$-Cartier.

    $K_Y + \Gamma_Y + 2d A_Y \sim \underset{<0.}{\underline{(a' - \varepsilon')}} T_Y / X \quad \underline{\text{ample } / X}$

  - $C \subset Y$ curve not contracted by ϕ $\quad c \in C$

  Step 2.&3 $\begin{cases} K_{Y'} + \Gamma_{Y'} + 2d A_{Y'} \sim_{\mathbb{R}} G_{Y'} / X \quad \text{semiample.} \\ G_{Y'} \text{ supported over finitely closed pts on } X. \end{cases}$

  $\Rightarrow \exists\, 0 \le P_{Y'} \sim_{\mathbb{R}} K_{Y'} + \Gamma_{Y'} + 2d A_{Y'}$ s.t.

  $P_{Y'}$ does not intersect fiber of ϕ' over $\phi(c)$

  i.e. $\phi'_* P_{Y'} \not\ni \phi(c)$.

  Step 4 $\quad Y \xrightarrow{\phi} X$ is isom. over $\phi(c) \Rightarrow P_Y \not\ni c$.

  $\Rightarrow 0 \le P_Y \cdot C = (K_Y + \Gamma_Y + 2d \cdot A_Y) \cdot C$

  $\Rightarrow \underline{K_Y + \Gamma_Y + 2d A_Y} \quad \underline{\text{nef}} \ \& \ \underline{\text{ample } / X}$

  $A_Y = \phi^* A \quad A$ ample on X

  $\Rightarrow K_Y + \Gamma_Y + 3d A_Y$ is ample.

  Step 6: $\exists\, l = l(d, r, \varepsilon) \in \mathbb{Z}_{>0}$ s.t. $\underline{lA_Y - (K_Y + (1 - \varepsilon')T_Y)}$ $\underline{\text{is ample.}}$

  Write. $K_Y + \widetilde{B}_Y + t\widetilde{L}_Y + (1 - \varepsilon')T_Y = \phi^*(K_X + B + tL)$

  $lA_Y - (K_Y + (1 - \varepsilon')T_Y) =$

  $\boxed{(l - \tfrac{3d}{v})A_Y - (1 + \tfrac{1}{v})\,\phi^*(K_X + B + tL)}$ ①

  $+ \boxed{\tfrac{1}{v}\phi^*(K_X + B + tL) + \widetilde{B}_Y + t\widetilde{L}_Y + \tfrac{3d}{v}A_Y}$ ②

  Step 2&5 ② $= \frac{1}{v}(K_Y + \Gamma_Y + 3d A_Y)$ $\quad \underline{\text{ample.}}$

  A very ample $\quad A^d \le r \Rightarrow \exists\, e' = e'(d, r) \in \mathbb{Z}_{>0}$ s.t.

  $e'A - K_X$ ample.

  $A - B \quad A - L$ ample

  $\Rightarrow \exists\, l = l(d, r, \varepsilon)$ s.t.

  ① $= \phi^*\underset{\text{ample on } X}{\underline{((l - \tfrac{3d}{v})A - (1 + \tfrac{1}{v})(K_X + B + tL))}}$

  nef.

  $\Rightarrow lA_Y - (K_Y + (1 - \varepsilon')T_Y)$ is ample.

  Step 7 $\quad \underline{lA_Y - (K_Y + T_Y)}$ ample.

  by computation & Step 5.

  Step 8. Finish the proof by Thm 1.9.

  - $(Y, (1 - \varepsilon')T_Y)$ klt $\quad$ crepant to $(X, B + tL)$.

    $\Rightarrow (Y, T_Y)$ lc.

  - $A_Y|_{T_Y} \sim 0 \quad T_Y$ mapped to a closed pt. x

  Apply Thm 1.9 $\quad M = lA_Y$. $S = T_Y$ $\quad z = x$.

  $\boxed{\begin{array}{l} \exists\, n = n(d) \text{ and } \mathbb{Q}\text{-div } \Lambda_Y \ge T_Y \text{ s.t.} \\ (Y, \Lambda_Y) \text{ lc over } x \text{ and} \\ n(K_Y + \Lambda_Y) \sim (n + 2)\, lA_Y. \end{array}}$

  $\Rightarrow_{①} n\Lambda$ integral.

  $\Rightarrow 4lA_Y - (K_Y + \Lambda_Y)$ is ample. $\Rightarrow \exists\, m$ s.t. $mA - \Lambda$ ample

  $\Lambda = \phi_* \Lambda_Y \qquad K_Y + \Lambda_Y = \phi^*(K_X + \Lambda)$

  $\Rightarrow_{②} mA - \Lambda$ ample.

  ③ (X, Λ) lc near x

  ④ $a(T, X, \Lambda) = a(T, Y, \Lambda_Y) = a(T_Y, Y, \Lambda_Y) = 0$

- Proof of Thm 1.8 (6 steps).

  Step 1: $\underline{\operatorname{lct}(X, B; |M|_{\mathbb{R}}) \ge \operatorname{lct}(X, B; |A|_{\mathbb{R}})}$.

  $A - B$. $\quad A - M \quad$ ps-eff. $\quad \Uparrow$

  WMA $\quad$ ~ $\quad$ big $\quad A \sim_{\mathbb{R}} M + N \quad N \ge 0$.

  STS $\quad \operatorname{lct}(X, B; |A|_{\mathbb{R}})$ has a lower bound.

  Step 2: $\underline{\text{WMA } K_X \ \mathbb{Q}\text{-Cartier.}}$

  Lem 2.24 $\quad \exists\, l = l(d, r, \varepsilon) \quad Y \xrightarrow{\phi} X \quad A_Y$ v.a. on Y
  - Y normal $\quad K_Y$ $\mathbb{Q}$-Cartier
  - $A_Y^d \le l$

  Let $K_Y + B_Y = \phi^*(K_X + B)$

  $A_Y - B_Y = \boxed{\underset{\text{ample.}}{\underline{(A_Y - \phi^* A)}}} + \underset{\text{ps eff.}}{\underline{(\phi^* A - B_Y)}}$ big.

  $\operatorname{lct}(X, B; |A|_{\mathbb{R}}) = \operatorname{lct}(Y, B; |\phi^* A|_{\mathbb{R}})$

  $\ge \operatorname{lct}(Y, B; |A_Y|_{\mathbb{R}})$

  $(X, B), A, r \quad \rightsquigarrow \quad (Y, B_Y), A_Y, l$

  Step 3 $\quad \underline{\text{WMA } A - B \text{ ample.}}$ $\quad (Y. X_u)$

  Step 1. $A - B \sim_{\mathbb{R}} P \ge 0$

  apply thm w/ $B = 0$ $\quad \exists\, s = s(d, r, \varepsilon)$ s.t. (X, sP) klt.

  $\Rightarrow K_X + sP + 3d A$ ample.

  $(e'A - K_X$ ample$)$

  $\Rightarrow sP + (3d + e')A$ ample.

  $e := \left\lceil \frac{3d + e'}{s} \right\rceil + 1$

  $\Rightarrow \underline{e}A - B \sim_{\mathbb{R}} \left\lceil \frac{3d + e'}{s} \right\rceil A + P$ ample

  From now on assume $A - B$ ample.

  Step 4: $\quad C = \frac{1}{2}A$

  $\operatorname{lct}(X, B; |A|_{\mathbb{R}}) = \frac{1}{2}\operatorname{lct}(X, B; |C|_{\mathbb{R}})$

  STS : $\operatorname{lct}(X, B; |C|_{\mathbb{R}})$ has a lower bound. > 0

  5.11 $\quad n, m.\ \varepsilon'$

  $t = \varepsilon'$-lct of $(X, B; L)$ $\quad L \in |C|_{\mathbb{R}}$

  STS $\quad t$ has a lower bound > 0 $\quad$ WMA $t < 1$

  $\exists\, T / X$ s.t. $a(T, X, B + tL) = \varepsilon'$

  Let x be the generic pt of $c_X(T)$.

  Step 5. $\underline{\text{WMA } x \in X \text{ is a closed pt}}$

  If not. cutting by $H \in |A|$ general.

  $\exists\, v = v(d, r, \varepsilon) \in \mathbb{R}_{>0}$ s.t. $(X, B + vL)$ is lc.

  outside finitely many closed pts. in particular near x.

  Then $(X, B + (1 - \frac{\varepsilon'}{\varepsilon})vL) \quad \varepsilon'$-lc near x.

  $\Rightarrow \overset{t=}{\operatorname{lct}} \ge (1 - \frac{\varepsilon'}{\varepsilon})v$

  Step 6 $\quad \underline{t \text{ has a lower bound}}$

  5.11 $\quad \exists$ $\mathbb{Q}$-Cart $\mathbb{Q}$-div $\Lambda \ge 0$ s.t.

  4 conditions hold.

  WMA $\quad A - B - tL \quad A - \Lambda$ ample.

  5.7 to $(X, B + tL)$

  $\exists\, q = q(d, r, n, \varepsilon') \in \mathbb{Z}_{>0}$ s.t.

  if $T \subset U \xrightarrow{\psi} X$ $\quad$ then $\operatorname{mult}_E \psi^* L \le q$

  $\underline{a(T, X, B) \ge \varepsilon > \varepsilon' = a(T, X, B + tL)}$

  $\operatorname{mult}_T \psi^* tL = t \operatorname{mult}_E \psi^* L \ge \varepsilon - \varepsilon'$

  $\Rightarrow t \ge \frac{\varepsilon - \varepsilon'}{q} \qquad \varepsilon' = \varepsilon'(d, r)$ $\quad \square$

- Pf of 1.1 (BAB)
  - Reduce to $B = 0$
  - eff. bir. bdd. of complement 1.1_{d-1} $\Big\} \Rightarrow \operatorname{vol}(-K_X) \le v$ $\quad$ Sect 2. $\Rightarrow 1.1_d$.

  $1.8 \Rightarrow 1.6_d \Rightarrow \alpha \ge t$ $\quad \square$

- Th'm 1.7 : If $\alpha \le 1$ Then $\exists\, D \sim_{\mathbb{R}} -(K_X + B)$

  s.t. $\operatorname{lct}(|-(K_X + B)|_{\mathbb{R}}) = \operatorname{lct}(D)$

  Thm 1.8.

  + Sect 3. $\quad \square$